JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 5, September–October 1993

Engineering Notes

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Asymptotic Stability Theorem for Autonomous Systems

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I. Introduction

THERE exist Lyapunov stability theorems[1] for proving the uniform asymptotic stability of autonomous and nonautonomous dynamical systems. For autonomous systems, the asymptotic stability of an equilibrium point can be shown by constructing a Lyapunov function V in the neighborhood of the equilibrium point whose derivative $\dot{V}$ is negative definite. For nonautonomous systems, the uniform asymptotic stability is guaranteed by imposing an additional constraint—the Lyapunov function has to be decrescent. In reality, we very often come across Lyapunov functions whose derivatives are only negative semidefinite. For autonomous systems it may then still be possible to conclude the asymptotic stability using LaSalle's invariant set theorem,[2] provided we can show that the maximum invariant set contains only the equilibrium point. It is easy and always possible to identify the set of points where the derivative of the Lyapunov function vanishes, but the maximum invariant set is only a subset of this set. The main challenge of LaSalle's theorem is therefore to sort out the maximum invariant set. This task may not be simple for complex systems.

For the general class of nonautonomous systems, invariant set theorems do not exist. This is because the positive limit set of the trajectories of nonautonomous systems are not invariant sets in general. Naturally, the asymptotic stability analysis of nonautonomous systems is more difficult. This difficulty is often overcome by the use of Barbalat's lemma.[3] When applied to a Lyapunov function V whose derivative is negative semidefinite and uniformly continuous, Barbalat's lemma predicts the convergence of $\dot{V}$ to zero asymptotically. Though this lemma does not establish the convergence of V to zero, it often leads to satisfactory solution of control problems, as in the case of adaptive control.

In the next section we develop a theorem that provides us with sufficient conditions for proving the asymptotic stability of autonomous and periodic nonautonomous systems in the event where the first derivative of the Lyapunov function is zero. We assume an analytic Lyapunov function whose higher-order derivatives can be easily computed. When the first derivative of the Lyapunov function vanishes at points other than at the equilibrium point, only the higher-order derivatives of this nonincreasing function can tell us whether the function reduces further or remains at its constant value for all future times. The sufficient conditions of our theorem involve these higher-order derivatives that essentially contain more information of the system dynamics. These conditions allow us to effectively sort out the maximum invariant set from the set of points where the first derivative of the Lyapunov function vanishes. Our theorem is more versatile than the well-known LaSalle's theorem[2] because it does not require us to sort out the maximum invariant set. We demonstrate the efficacy of our theorem through two simple examples.

II. Theorem on Asymptotic Stability

Before stating our asymptotic stability theorem, we state two simple lemmas that provide a basis for our theorem.

Lemma 1: A real function $f(t) \in C^2$ defined in (a,b) is concave if and only if $f(t)'' \leq 0$, $\forall x \in (a,b)$.

Proof: 1) Necessity: Let $x \in (a,b)$. Then for h small enough, $x-h$, $x+h \in (a,b)$. From the definition of concavity,[4] $f(x) \geq \frac{1}{2}[f(x-h)+f(x+h)]$. Therefore, because $f \in C^2$

$$f''(x) = \lim_{h \to 0} \frac{f(x-h)+f(x+h)-2f(x)}{h^2} \leq 0 \tag{1}$$

2) Sufficiency: Let $x,y \in (a,b)$, and $x<y$. For $\lambda \in [0, 1]$ and $t = \lambda x + (1-\lambda)y$, the first-order Taylor's series approximation of $f(x)$ and $f(y)$ are, respectively

$$f(x) = f(t) + f'(t)(x-t) + f''(\xi_1)(x-t)^2, \quad \xi_1 \in [x,t]$$

$$f(y) = f(t) + f'(t)(y-t) + f''(\xi_2)(y-t)^2, \quad \xi_2 \in [t,y] \tag{2}$$

Therefore, it follows that

$$\lambda f(x) + (1-\lambda)f(y) = f(t) + \lambda f''(\xi_1)(x-t)^2 + (1-\lambda)f''(\xi_2)(y-t)^2 \leq f(t) \quad \text{since } f''(\xi_1),\ f''(\xi_2) \leq 0 \tag{3}$$

Therefore, the function is concave by definition. □

Lemma 2: Let $f(t)$ be a nonpositive function such that $f(t_0)=0$ and $f(t)<0$ for some values of t. If the function $f(t)$ is analytic, then $f(t)$ is concave in some open neighborhood of t_0.

Proof: Because the function $f(t)$ is analytic, all derivatives of the function exist, and the function can be expanded using Taylor's series as

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n \tag{4}$$

Let us next assume that the function $f(t)$ is not concave in any open neighborhood of t_0. This implies from Lemma 1 that the condition $f''(t) \leq 0$ does not hold good in any open neighborhood of t_0. Therefore, either $f''(t) \geq 0$ or $f''(t)$ changes sign in every open neighborhood of t_0. If $f''(t) \geq 0$ in every open neighborhood of t_0, then we can show from the corollary of

Received April 15, 1992; revision received Aug. 24, 1992; accepted for publication Sept. 2, 1992.

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Lemma 1 that $f(t)$ is convex everywhere. This is not true because $f(t)$ is nonpositive and has a maximum value at $t = t_0$. The other possibility is that $f''(t)$ changes sign in every open neighborhood of t_0. Then $f^{(n)}(t)$, for $n = 2, 3, \ldots, \infty$, also changes sign in every open neighborhood of t_0. This implies that $f^{(n)}(t_0) = 0$, for $n = 2, 3, \ldots, \infty$. Now, since $f(t)$ is nonpositive and $f(t_0) = 0$, $f(t)$ achieves a local maximum at t_0. Therefore, $f'(t_0) = 0$. Substituting these results in Eq. (4), we have $f(t) = 0$. This cannot be true because $f(t)$ is strictly negative for some values of t. We have therefore proved by contradiction that $f(t)$ is concave in some open neighborhood of t_0. □

Using the two lemmas stated previously we can conclude that if $f(t)$ is an analytic nonpositive function, and if $f(t_0) = 0$, then $f'(t_0) = 0$ simply because $f(t)$ is locally maximum at t_0, and $f''(t)$ is concave in some neighborhood of t_0, or $f''(t) \leq 0$ in some open neighborhood of t_0. Additionally if $f''(t_0) = 0$, then we can apply our lemmas to $f''(t)$, which is itself an analytical nonpositive function. In that case $f'''(t_0) = 0$ and $f''''(t) \leq 0$ in some open neighborhood of t_0. Clearly, the lemmas can be applied recursively, and we can conclude the following: When some even derivative of $f(t)$ vanishes at t_0, the next higher derivative that is an odd derivative also vanishes at t_0, and the second next derivative is nonpositive in some open neighborhood of t_0.

We now consider the nonautonomous system

$$\dot{x} = f\big(t, x(t)\big) \tag{5}$$

where $f : R_+ \times D \to R^n$ is a smooth vector field on $R_+ \times D$; $D \subset R^n$ is a neighborhood of the origin $x = 0$. Let $x = 0$ be an equilibrium point for the system described by Eq. (5). We then have

$$f(t, 0) = 0, \qquad \forall t \geq 0 \tag{6}$$

We next state our theorem on asymptotic stability.

Theorem: 1) A necessary condition for stable nonautonomous systems: Let $V(t,x) : R_+ \times D \to R_+$ be locally positive definite and analytic on $R_+ \times D$, such that

$$\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right) f(t,x) \tag{7}$$

is locally negative semidefinite. Then whenever an odd derivative of V vanishes, the next derivative necessarily vanishes and the second next derivative is necessarily negative semidefinite. 2) A sufficient condition for asymptotically stable autonomous systems: Let $V(x) : D \to R_+$ be locally positive definite and analytic on D, such that $\dot{V} \leq 0$. If there exists a positive integer k such that

$$\begin{cases} V^{(2k+1)}(x) < 0 & \forall x \neq 0 : \dot{V}(x) = 0 \\ V^{(i)}(x) = 0 & \text{for } i = 2, 3, \ldots, 2k \end{cases} \tag{8}$$

where $V^{(*)}(x)$ denotes the (*)-th time derivative of V with respect to time, then the system is asymptotically stable. However, if $V^{(j)}(x) = 0$, $\forall j = 1, 2, \ldots, \infty$, then the sufficient condition for the autonomous system to be asymptotically stable is that the set

$$S = \left\{x : V^{(j)}(x) = 0, \quad \forall j = 1, 2, \ldots, \infty\right\}$$

contains only the trivial trajectory $x = 0$.

Proof: Part 1 of the theorem can be proven very easily with the help of Lemmas 1 and 2.

To prove part 2, we first realize that $x = 0$ is stable by standard argument because V is locally positive definite and $\dot{V} \leq 0$.

Next, since V is bounded from below by zero and V is nonincreasing ($\dot{V} \leq 0$), $V \to \alpha$, $\alpha \geq 0$, as $t \to \infty$.

Because V is analytic and therefore smooth, $\dot{V}$ is uniformly continuous. Hence when $V \to \alpha$, $\dot{V} \to 0$ as $t \to \infty$, by Barbalat's lemma.

Since V is locally positive definite, $V \to 0$ implies $x \to 0$ as $t \to \infty$. Therefore, if we can show that $\alpha = 0$ we can conclude asymptotic stability. We prove $\alpha = 0$ by contradiction. Since $V \to \alpha \neq 0$ and V is locally positive definite, $\exists$ an open neighborhood N of $x = 0$ such that the trajectory of $x(t)$ lies outside $N \forall t > T$, and for some $T \geq 0$.

Let $Q = \{x : \dot{V}(x) = 0\}$. Since $x(t)$ converges to Q but lies outside N for large t, the set $W = Q - N$ is nonempty and is the limit set for $x(t)$. If the conditions given by Eq. (8) hold, then

$$\begin{gathered} V^{(i)}(x) = 0 \qquad \forall i = 1, 2, \ldots, 2k, \qquad \forall x \in W \\ \max_{x \in W} V^{(2k+1)}(x) = -\gamma < 0 \end{gathered} \tag{9}$$

Pick an ϵ arbitrarily small. Since V is analytic and therefore all of its derivatives are continuous, $\exists$ an open neighborhood U of W whose closure U^C does not contain $x = 0$ and $\forall x \in U^C$:

$$\begin{gathered} |V^{(i)}(x)| \leq \epsilon, \qquad \forall i = 1, 2, \ldots, 2k \\ V^{(2k+1)}(x) \leq -\gamma + \epsilon \end{gathered} \tag{10}$$

Since $x(t) \to W$ as $t \to \infty$, $\exists T_1$ such that $x(t) \in U^C \forall t \geq T_1$. Now integrating $V^{(2k+1)}(t)$ with respect to time to get V, we have

$$\begin{aligned} V(t) - V(T_1) &= \underbrace{\int_{T_1}^{t} \cdots \int_{T_1}^{t}}_{(2k+1)} V^{(2k+1)}(t)\, dt \\ &\leq \underbrace{\int_{T_1}^{t} \cdots \int_{T_1}^{t}}_{(2k+1)} -(\gamma - \epsilon)\, dt \\ &= -(\gamma - \epsilon)\frac{(t - T_1)^{2k+1}}{(2k+1)!} \\ &\quad + \epsilon \frac{(t - T_1)^{2k}}{(2k)!} + \epsilon \frac{(t - T_1)^{2k-1}}{(2k-1)!} \\ &\quad + \cdots + \epsilon(t - T_1) \triangleq \delta(t) \end{aligned} \tag{11}$$

Hence, $V(t) \leq V(T_1) + \delta(t)$, where $\delta(t) \to -\infty$ as $t \to \infty$, since $\gamma > 0$ and ϵ is arbitrarily small. Since $V(T_1) \leq V(t = 0)$ and is therefore bounded, $V(t) \to -\infty$ as $t \to \infty$. This contradicts the fact that $V \geq 0$. Hence $\alpha = 0$, and that implies that the system is asymptotically stable.

If all the derivatives of V are zero simultaneously, then the set $S = \{x : V^{(j)}(x) = 0, \forall j = 1, 2, \ldots, \infty\}$ is obviously an invariant set. Therefore, if S contains only the trivial trajectory, the autonomous system will be asymptotically stable. □

The idea behind the theorem discussed previously is simple. Consider an analytic function of time, whose functional dependent on time may be explicit or implicit. The behavior of such a function, at any time $t = T$, $T \in [0, \infty)$, can be predicted at time $t = 0$ from the behavior of the derivatives of the function at time $t = 0$. In our theorem we assumed an analytic Lyapunov function, which is a nonincreasing function by definition. If the first derivative of this Lyapunov function goes to zero at time $t = t_0$, the function remains constant at the instant of time $t = t_0$. The only way to predict at time $t = t_0$ whether the Lyapunov function decreases at time $t \geq t_0$ is to evaluate its derivatives at time $t = t_0$. Our theorem provides the sufficient conditions involving the derivatives of the Lyapunov function to guarantee the asymptotic stability of the system.

Our theorem can be easily extended to periodic nonautonomous systems by making use of the following theorem.[5,6]

Theorem: Consider the system given by Eq. (5) and suppose f satisfies

$$f(t,x) = f(t + T, x), \qquad \forall x \in R^n, \qquad \forall t \geq 0$$

for some positive number T. Then, the following two statements are equivalent:

1) The equilibrium point 0 of the system given by Eq. (5) is asymptotically stable at some time $t_0 \geq 0$.

2) The equilibrium point 0 of the system given by Eq. (6) is uniformly asymptotically stable over the interval $[0,\infty)$.

III. Examples

In this section we illustrate the efficacy of our theorem through two simple examples.

Example 1

Consider the simple example of the single-degree-of-freedom mass spring damper system

$$m\ddot{x} + c\dot{x} + kx = 0, \qquad m,c,k > 0 \tag{12}$$

where m, c, and k are the mass, damping, and stiffness of the system. We consider the analytic Lyapunov function V where

$$V = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 \tag{13}$$

represents the total energy of the system and whose derivative is given by the relation $\dot{V} = -c\dot{x}^2 \leq 0$. We compute the higher-order derivatives of V and check that when $\dot{V} = 0$, $\ddot{V} = 0$, and $V^{(3)} = -2c\ddot{x}^2 = -2c(k/m)^2 x^2 < 0$, $\forall x \neq 0$, where $V^{(3)}$ is the third derivative of V with respect to time. Clearly, from the sufficient conditions of our theorem, we can conclude the asymptotic stability of the equilibrium point $x = 0$ of the system described by Eq. (12).

Example 2

Consider the damped Mathieu equation that represents a nonautonomous system with a period of 2π. The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_2 - (2 + \sin t)x_1 \tag{14}$$

We choose the Lyapunov function V as

$$V(t, x_1, x_2) = x_1^2 + \frac{x_2^2}{2 + \sin t} \tag{15}$$

which is analytic in x_1, x_2, and t. Calculating $\dot{V}$, we find that

$$\dot{V} = -x_2^2 g(t), \qquad g(t) \triangleq \frac{4 + 2\sin t + \cos t}{(2 + \sin t)^2} \tag{16}$$

Thus $\dot{V} \leq 0$ for all t, (x_1, x_2) and $\dot{V} = 0$ if and only if $x_2 = 0$. We compute the higher-order derivatives of V and find that when $\dot{V} = 0$ or $x_2 = 0$, then $\ddot{V} = 0$ and

$$V^{(3)} = -2\dot{x}_2^2 g(t) = -2(2 + \sin t)^2 x_1^2 g(t) < 0$$

$$\forall \boldsymbol{x} \triangleq (x_1 x_2)^T \neq 0 \tag{17}$$

It therefore follows from our theorem that the system described by Eq. (14) is uniformly asymptotically stable in the neighborhood of the equilibrium point.

IV. Conclusion

An asymptotic stability theorem for autonomous systems and periodic nonautonomous systems was developed that provides sufficient conditions to conclude asymptotic stability when the first derivative of the Lyapunov function vanishes. This theorem is more versatile than the well-known LaSalle's theorem because it does not require us to sort out the maximum invariant set. The theorem is applicable to analytic Lyapunov functions and is specially useful when higher-order derivatives of the Lyapunov function are easy to compute. The efficacy of our theorem was shown through two simple examples.

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Exact Closed-Form Solution of Generalized Proportional Navigation

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I. Introduction

PROPORTIONAL navigation has been widely used as the guidance scheme in the homing phase of flight for most missile systems. In pure proportional navigation (PPN), the commanded acceleration is applied in the direction normal to pursuer's velocity, and its magnitude is proportional to the angular rate of line of sight (LOS) between pursuer and its target.[1-4] In traditional true proportional navigation (TPN), the commanded acceleration is applied in a direction normal to the LOS, and its magnitude is proportional to the LOS rate.[5,6] Then a modified TPN is submitted, in which the commanded acceleration is applied in a direction normal to the LOS, and its magnitude is proportional to the product of LOS rate and closing speed between pursuer and target.[7,8] Furthermore, generalized proportional navigation (GPN) and ideal proportional navigation (IPN) were presented recently, in which the commanded acceleration is applied with a fixed bias angle to the direction normal to LOS and normal to the relative velocity between pursuer and target, respectively.[9-11]

In GPN, some solutions were previously obtained for a nonmaneuvering target that seem incomplete.[9,10] In this Note, we try to derive an exact and complete closed-form solution of GPN with a maneuvering target, which is much more general and comprehensive than those obtained before. Then a special case of target maneuver is discussed to easily illustrate the effect of a target maneuver. It can be solved as a function of deflection angle of LOS, in general. Some important and significant characteristics related to the system performance, such as capture capability and energy cost, are investigated and discussed in detail.

II. Closed-Form Solution

Consider a pursuer of speed V_M and a maneuvering target with speed V_T in exoatmospheric flight under the guidance law of GPN. The commanded acceleration is given with a bias angle β to the direction normal to LOS and its magnitude is proportional to LOS rate, i.e.,

$$\boldsymbol{a}_c = -\lambda v_0 \dot{\theta}(\cos\beta \boldsymbol{e}_\theta + \sin\beta \boldsymbol{e}_r) \tag{1}$$

Received March 6, 1992; revision received Aug. 2, 1992; accepted for publication Dec. 26, 1992.

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